SIMPSON TYPE INEQUALITIES VIA $\varphi-$CONVEXITY

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Abstract. In this paper, we obtain some Simpson type inequalities for functions whose derivatives in absolute value are $\varphi-$convex.

1. introduction and preliminaries

Suppose $f:[a,b]\to\mathbb{R}$ is a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_{\infty}=\sup\left|f^{(4)}(x)\right|<\infty$. The following inequality

$$\begin{aligned} &\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ &\frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^4 \end{aligned}$$

is well known in the literature as Simpson's inequality.

For some results about Simpson inequality see [3]-[7].

In [3], Alomari et al. proved some inequalities of Simpson type for $s-$convex functions by using the following Lemma.

Lemma 1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be an absolutely continuous mapping on I° where $a,b\in I$ with $a<b$. Then the following equality holds:*

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ =\ &(b-a)\int_0^1 p(t)f'(tb+(1-t)a)dt, \end{aligned}$$

where

$$p(t)=\begin{cases} t-\frac{1}{6}, & t\in\left[0,\frac{1}{2}\right) \\ t-\frac{5}{6}, & t\in\left[\frac{1}{2},1\right]. \end{cases}$$

Let $f,\varphi:K\to\mathbb{R}$, where K is a nonempty closed set in $\mathbb{R}^n$, be continuous functions. We recall the following results, which are due to Noor and Noor [1], Noor [2] as follows:

Definition 1. *Let $u\in K$. Then the set K is said to be $\varphi-$convex at u with respect to φ, if*

$$u+te^{i\varphi}(v-u)\in K,\quad \forall u,v\in K,\quad t\in[0,1].$$

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Remark 1. *We would like to mention that the Definition 1 of a $\varphi-$convex set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point u which is contained in K. We don't require that the point v should be one of the end points of the path. This observation plays an important role in our analysis . Note that, if we demand that v should be an end point of the path for every pair of points, $u, v \in K$, then $e^{i\varphi}(v-u) = v-u$ if and only if, $\varphi = 0$, and consequently $\varphi-$convexity reduces to convexity. Thus, it is true that every convex set is also an $\varphi-$convex set, but the converse is not necessarily true.*

Definition 2. *The function f on the $\varphi-$convex set K is said to be $\varphi-$convex with respect to φ, if*

$$f(u+te^{i\varphi}(v-u)) \leq (1-t)f(u)+tf(v), \quad \forall u,v \in K, \quad t\in[0,1].$$

The function f is said to be $\varphi-$concave if and only if $-f$ is $\varphi-$convex. Note that every convex function is a $\varphi-$convex function, but the converse is not true.

The following inequality is known as the Hölder inequality[8]:

Theorem 1. *Let $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on $[a,b]$ and if $|f|^p$ and $|g|^q$ are integrable functions on $[a,b]$ then*

$$\int_a^b |f(x)g(x)|\,dx \leq \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\,dx\right)^{\frac{1}{q}},$$

with equality holding if and only if $A\,|f(x)|^p = B\,|g(x)|^q$ almost everywhere, where A and B are constants.

2. Simpson type inequalities for $\varphi-$convex functions

Throughout this section, let $K=[a,a+e^{i\varphi}(b-a)]$ and $0\leq\varphi\leq\frac{\pi}{2}$

We used the following Lemma to obtain our main results.

Lemma 2. *Let $K\subset\mathbb{R}$ be a $\varphi-$convex subset and $f:K\to(0,\infty)$ be a differentiable function on K°(the interior of K), $a,b\in K$ with $a<a+e^{i\varphi}(b-a)$. If f' is integrable on $[a,a+e^{i\varphi}(b-a)]$, following equality holds:*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)+f(a+e^{i\varphi}(b-a))\right]-\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx\right| \\ =\;& e^{i\varphi}(b-a)\int_0^1 p(t)f'(a+te^{i\varphi}(b-a))dt,\end{aligned}$$

where

$$p(t)=\begin{cases} t-\frac{1}{6}, & t\in\left[0,\frac{1}{2}\right) \\ t-\frac{5}{6}, & t\in\left[\frac{1}{2},1\right]. \end{cases}$$

Proof. Since K is a $\varphi-$convex set, for $a,b\in K$ and $t\in[0,1]$ we have $a+e^{i\varphi}(b-a)\in K$. Integrating by parts implies that

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left(t-\frac{1}{6}\right)f'(a+te^{i\varphi}(b-a))dt+\int_{\frac{1}{2}}^1\left(t-\frac{5}{6}\right)f'(a+te^{i\varphi}(b-a))dt\\
=&\left(t-\frac{1}{6}\right)\frac{f(a+te^{i\varphi}(b-a))}{e^{i\varphi}(b-a)}\Bigg|_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}}\frac{f(a+te^{i\varphi}(b-a))}{e^{i\varphi}(b-a)}dt\\
&+\left(t-\frac{5}{6}\right)\frac{f(a+te^{i\varphi}(b-a))}{e^{i\varphi}(b-a)}\Bigg|_{\frac{1}{2}}^1-\int_{\frac{1}{2}}^1\frac{f(a+te^{i\varphi}(b-a))}{e^{i\varphi}(b-a)}dt\\
=&\frac{1}{6e^{i\varphi}(b-a)}\left[f(a)+4f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)+f(a+e^{i\varphi}(b-a))\right]\\
&-\frac{1}{e^{i\varphi}(b-a)}\left[\int_0^{\frac{1}{2}}f(a+te^{i\varphi}(b-a))dt+\int_{\frac{1}{2}}^1 f(a+te^{i\varphi}(b-a))dt\right].
\end{aligned}$$

If we change the variable $x=a+te^{i\varphi}(b-a)$ and multiply the resulting equality with $e^{i\varphi}(b-a)$ we get the desired result. □

Theorem 2. *Let $f:K\to(0,\infty)$ be a differentiable function on K°. If $|f'|$ is $\varphi-$convex function on K° and $a,b\in K$ with $a<a+e^{i\varphi}(b-a)$. Then , the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)+f(a+e^{i\varphi}(b-a))\right]-\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)}f(x)dx\right|\\
\leq&\frac{5}{72}e^{i\varphi}(b-a)\left[|f'(a)|+|f'(b)|\right].
\end{aligned}$$

Proof. From Lemma 2 and using the $\varphi-$convexity of $|f'|$ we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)+f(a+e^{i\varphi}(b-a))\right]-\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)}f(x)dx\right|\\
\leq&e^{i\varphi}(b-a)\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|f'(a+te^{i\varphi}(b-a))\right|dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left|f'(a+te^{i\varphi}(b-a))\right|dt\right\}\\
\leq&e^{i\varphi}(b-a)\left\{\int_0^{\frac{1}{6}}\left(\frac{1}{6}-t\right)\left[(1-t)|f'(a)|+t|f'(b)|\right]dt\right.\\
&+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)\left[(1-t)|f'(a)|+t|f'(b)|\right]dt\\
&+\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)\left[(1-t)|f'(a)|+t|f'(b)|\right]dt\\
&+\int_{\frac{5}{6}}^1\left(t-\frac{5}{6}\right)\left[(1-t)|f'(a)|+t|f'(b)|\right]dt\\
=&\frac{5}{72}e^{i\varphi}(b-a)\left[|f'(a)|+|f'(b)|\right]
\end{aligned}$$

which completes the proof. □

Theorem 3. *Let $f : K \to (0,\infty)$ be a differentiable function on K°, $a,b \in K$ with $a < a+e^{i\varphi}(b-a)$. If $|f'|^q$ is $\varphi-$convex function on K° for some fixed $q>1$ then the following inequality holds*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)+f(a+e^{i\varphi}(b-a))\right]-\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)}f(x)dx\right| \\
\leq\ & e^{i\varphi}(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} \\
&\times\left\{\left(\frac{3}{8}\left|f'(a)\right|^q+\frac{1}{8}\left|f'(b)\right|^q\right)^{\frac{1}{q}}+\left(\frac{1}{8}\left|f'(a)\right|^q+\frac{3}{8}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\}
\end{aligned}
$$

where $p=\frac{q}{q-1}$.

Proof. From Lemma 2 and using the Hölder inequality, we have

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)+f(a+e^{i\varphi}(b-a))\right]-\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)}f(x)dx\right| \\
\leq\ & e^{i\varphi}(b-a)\left\{\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'(a+te^{i\varphi}(b-a))\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'(a+te^{i\varphi}(b-a))\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Since $|f'|^q$ is $\varphi-$convex, we obtain

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)+f(a+e^{i\varphi}(b-a))\right]-\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)}f(x)dx\right| \\
\leq\ & e^{i\varphi}(b-a)\left\{\left(\int_0^{\frac{1}{6}}\left(\frac{1}{6}-t\right)^p dt+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)^p dt\right)^{\frac{1}{p}}\right. \\
&\times\left(\int_0^{\frac{1}{2}}\left[(1-t)\left|f'(a)\right|^q+t\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)^p dt+\int_{\frac{5}{6}}^1\left(t-\frac{5}{6}\right)^p dt\right)^{\frac{1}{p}} \\
&\left.\times\left(\int_{\frac{1}{2}}^1\left[(1-t)\left|f'(a)\right|^q+t\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\} \\
=\ & e^{i\varphi}(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} \\
&\times\left\{\left(\frac{3}{8}\left|f'(a)\right|^q+\frac{1}{8}\left|f'(b)\right|^q\right)^{\frac{1}{q}}+\left(\frac{1}{8}\left|f'(a)\right|^q+\frac{3}{8}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\}
\end{aligned}
$$

which is the desired. □

Theorem 4. *Under the assumptions of Theorem 3, we have the following inequality*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)+f(a+e^{i\varphi}(b-a))\right]-\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)}f(x)dx\right| \\
\leq\ & e^{i\varphi}(b-a)\left(\frac{2(1+2^{p+1})}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

Proof. From Lemma 2, $\varphi-$convexity of $|f'|^q$ and using the Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)+f(a+e^{i\varphi}(b-a))\right]-\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)}f(x)dx\right| \\
\leq\ & e^{i\varphi}(b-a)\left[\int_0^1|p(t)|\left|f'(a+te^{i\varphi}(b-a))\right|dt\right] \\
\leq\ & e^{i\varphi}(b-a)\left(\int_0^1|p(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(a+te^{i\varphi}(b-a))\right|^q\,dt\right)^{\frac{1}{q}} \\
\leq\ & e^{i\varphi}(b-a)\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left[(1-t)\left|f'(a)\right|^q+t\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\
=\ & e^{i\varphi}(b-a)\left(\frac{2(1+2^{p+1})}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}
\end{aligned}$$

where we used the fact that

$$\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt=\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^p dt=\frac{(1+2^{p+1})}{6^{p+1}(p+1)}.$$

The proof is completed. □

Theorem 5. *Let $f: K\to(0,\infty)$ be a differentiable function on K°, $a,b\in K$ with $a<a+e^{i\varphi}(b-a)$. If $|f'|^q$ is $\varphi-$convex function on K° for some fixed $q\geq 1$ then the following inequality holds*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)+f(a+e^{i\varphi}(b-a))\right]-\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)}f(x)dx\right| \\
\leq\ & e^{i\varphi}(b-a)\left(\frac{5}{72}\right)^{1-\frac{1}{q}} \\
&\times\left\{\left(\frac{61\left|f'(a)\right|^q+29\left|f'(b)\right|^q}{1296}\right)^{\frac{1}{q}}+\left(\frac{29\left|f'(a)\right|^q+61\left|f'(b)\right|^q}{1296}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof. From Lemma 2 and using the power-mean inequality, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)+f(a+e^{i\varphi}(b-a))\right]-\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)}f(x)dx\right| \\
\leq\ & e^{i\varphi}(b-a) \\
&\times\left\{\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^{1-\frac{1}{q}}dt\right)^{\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|f'(a+te^{i\varphi}(b-a))\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^{1-\frac{1}{q}}dt\right)^{\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left|f'(a+te^{i\varphi}(b-a))\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Since $|f'|^q$ is $\varphi-$convex function we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|f'(a+te^{i\varphi}(b-a))\right|^q dt \\
\leq\ & \int_0^{\frac{1}{6}}\left(\frac{1}{6}-t\right)\left[(1-t)\left|f'(a)\right|^q+t\left|f'(b)\right|^q\right]dt \\
&+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)\left[(1-t)\left|f'(a)\right|^q+t\left|f'(b)\right|^q\right]dt \\
=\ & \frac{61\left|f'(a)\right|^q+29\left|f'(b)\right|^q}{1296}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left|f'(a+te^{i\varphi}(b-a))\right|^q dt \\
\leq\ & \int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)\left[(1-t)\left|f'(a)\right|^q+t\left|f'(b)\right|^q\right]dt \\
&+\int_{\frac{5}{6}}^1\left(t-\frac{5}{6}\right)\left[(1-t)\left|f'(a)\right|^q+t\left|f'(b)\right|^q\right]dt \\
=\ & \frac{29\left|f'(a)\right|^q+61\left|f'(b)\right|^q}{1296}.
\end{aligned}$$

Combining all the above inequalities gives us the desired result. □

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